SECTION 2

Objectives
- Identify appropriate coordinate systems for solving problems with vectors.
- Apply the Pythagorean theorem and tangent function to calculate the magnitude and direction of a resultant vector.
- Resolve vectors into components using the sine and cosine functions.
- Add vectors that are not perpendicular.

TEACH FROM VISUALS

FIGURE 2.1
Using a Coordinate System
A gecko’s displacement while climbing a tree can be represented by an arrow pointing along the $y$-axis.

FIGURE 2.2
Two Different Coordinate Systems
A plane traveling northeast at a velocity of 300 m/s can be represented as either (a) moving along a $y$-axis chosen to point to the northeast or (b) moving at an angle of 45° to both the $x$- and $y$-axes, which line up with west-east and south-north, respectively.

Teaching Tip
Review the sign conventions for coordinate systems that were established in the chapter “Motion in One Dimension.” Movements to the right along the $x$-axis and upward along the $y$-axis are considered positive, and movements to the left along the $x$-axis and downward along the $y$-axis are considered negative.

Teach

Preview Vocabulary
Scientific Meaning  The word simultaneous is used for phenomena that occur together at the same time. Ask students to list some simultaneous phenomena, such as the pressure of a confined gas decreasing while its volume is increasing.

FIGURE 2.2
Point out the two very different choices for coordinate axes.

Ask  Which set of axes will give the correct answer?

Answer: Either set of axes must give the same result.

Problem Solving

TAKE IT FURTHER
For Figure 2.2, ask students how the plane’s motion would be changed if there is a headwind of 10 m/s from the northeast. The plane’s velocity would be reduced to 290 m/s northeast. Invite a student to draw the vectors for this situation on the board, and to show classmates how to solve this problem both visually and mathematically.

Then ask, how would the situation be different with a tailwind of 10 m/s from the southwest? The plane’s velocity would be increased to 310 m/s northeast. What would be the total difference in velocity between a headwind of 10 m/s and a tailwind of 10 m/s?

20 m/s
Determining Resultant Magnitude and Direction

Earlier, we found the magnitude and direction of a resultant graphically. However, this approach is time-consuming, and the accuracy of the answer depends on how carefully the diagram is drawn and measured. A simpler method uses the Pythagorean theorem and the tangent function.

Use the Pythagorean theorem to find the magnitude of the resultant. Imagine a tourist climbing a pyramid in Egypt. The tourist knows the height and width of the pyramid and would like to know the distance covered in a climb from the bottom to the top of the pyramid. Assume that the tourist climbs directly up the middle of one face.

As can be seen in Figure 2.3, the magnitude of the tourist’s vertical displacement, \( \Delta y \), is the height of the pyramid. The magnitude of the horizontal displacement, \( \Delta x \), equals the distance from one edge of the pyramid to the middle, or half the pyramid’s width. Notice that these two vectors are perpendicular and form a right triangle with the displacement, \( d \).

As shown in Figure 2.4(a), the Pythagorean theorem states that for any right triangle, the square of the hypotenuse—the side opposite the right angle—equals the sum of the squares of the other two sides, or legs.

**Pythagorean Theorem for Right Triangles**

\[
c^2 = a^2 + b^2
\]

(length of hypotenuse)\(^2\) = (length of one leg)\(^2\) + (length of other leg)\(^2\)

In Figure 2.4(b), the Pythagorean theorem is applied to find the tourist’s displacement. The square of the displacement is equal to the sum of the square of the horizontal displacement and the square of the vertical displacement. In this way, you can find out the magnitude of the displacement, \( d \).

**Figure 2.3**

A Triangle Inside of a Pyramid

Because the base and height of a pyramid are perpendicular, we can find a tourist’s total displacement, \( d \), if we know the height, \( \Delta y \), and width, \( 2\Delta x \), of the pyramid.

**Figure 2.4**

Using the Pythagorean Theorem

(a) The Pythagorean theorem can be applied to any right triangle.
(b) It can also be applied to find the magnitude of a resultant displacement.

Misconception Alert!

Students often try to apply the Pythagorean theorem to triangles that do not contain a right angle. Point out that the Pythagorean theorem can be used only with a right triangle. Some students may know the Law of Cosines, which applies to all triangles. This law states that \( c^2 = a^2 + b^2 - 2ab\cos\theta \). The Law of Cosines can be used to calculate one side of any triangle when the opposite angle and the lengths of the other two sides are known. In this expression, \( c \) is the unknown side, \( \theta \) is the angle opposite \( c \), and \( a \) and \( b \) are the two known sides. Some students may attempt to use the Law of Cosines to add nonperpendicular vectors. This approach will give the correct answer, but it entails more computation and is more prone to student error when more than two vectors are to be added.

Teaching Tip

Point out that finding the resultant for the pyramid is fairly simple because the height, half-width, and hypotenuse form a right triangle. It is important to mention at this point that right triangles will also allow students to find the \( x \) and \( y \) components that are important for vector addition.

DECONSTRUCTING PROBLEMS

Emphasize the importance of recognizing what each symbol in an equation stands for. For example, in \( d^2 = \Delta x^2 + \Delta y^2 \), the \( \Delta \) is neither a variable nor a coefficient. In fact, both \( \Delta x \) and \( \Delta y \) are unique variables. That is, \( \Delta x \) or \( \Delta y \) can be replaced with only one value. Challenge students to investigate more about the symbol \( \Delta \), *delta*, and what it stands for in mathematics and physics.
Teach continued

Classroom Practice

Finding Resultant Magnitude and Direction A plane travels from Houston, Texas, to Washington, D.C., which is 1540 km east and 1160 km north of Houston. What is the total displacement of the plane?
Answer: 1930 km at 37.0° north of east

A camper travels 4.5 km northeast and 4.5 km northwest. What is the camper’s total displacement?
Answer: 6.4 km north

Teaching Tip

Explain that trigonometric functions such as the one in the text, \( \tan^{-1} \), have a different role. This is not an exponent. Instead, it is used to represent the inverse of a trigonometric function. For example, the inverse of the functions \( \sin x = 0.5 \) and \( \cos 3a = 0.92 \) take on the following forms:

\[ x = \sin^{-1} 0.5 = 30° \]

\[ 3a = \cos^{-1} 0.92 \Rightarrow a = \frac{\cos^{-1} 0.92}{3} = 7.7° \]

Problem Solving

TAKE IT FURTHER

Give students directions for a treasure hunt with movements that are all perpendicular to one another (i.e., move front/back and right/left). Have a student follow the directions to find a “treasure.” Then have the class resolve the directions into two components and calculate the resultant vector. Have a student use the resultant vector to go directly from the start position to the treasure.
Finding Resultant Magnitude and Direction (continued)

**PLAN**

Choose an equation or situation:
The Pythagorean theorem can be used to find the magnitude of the archaeologist's displacement. The direction of the displacement can be found by using the tangent function.

\[ d^2 = \Delta x^2 + \Delta y^2 \]

\[ \tan \theta = \frac{\Delta y}{\Delta x} \]

Rearrange the equations to isolate the unknowns:

\[ d = \sqrt{\Delta x^2 + \Delta y^2} \]

\[ \theta = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) \]

**SOLVE**

Substitute the values into the equations and solve:

\[ d = \sqrt{(115 \text{ m})^2 + (136 \text{ m})^2} \]

\[ d = 178 \text{ m} \]

\[ \theta = \tan^{-1}\left(\frac{136}{115}\right) \]

\[ \theta = 49.8^\circ \]

**CHECK YOUR WORK**

Because \( d \) is the hypotenuse, the archaeologist's displacement should be less than the sum of the height and half of the width. The angle is expected to be more than 45° because the height is greater than half of the width.

**Tips and Tricks**

Be sure your calculator is set to calculate angles measured in degrees. Some calculators have a button labeled “DRG” that, when pressed, toggles between degrees, radians, and grads.

### Practice A

1. A truck driver is attempting to deliver some furniture. First, he travels 8 km east, and then he turns around and travels 3 km west. Finally, he turns again and travels 12 km east to his destination.
   a. What distance has the driver traveled?
   b. What is the driver’s total displacement?

2. While following the directions on a treasure map, a pirate walks 45.0 m north and then turns and walks 7.5 m east. What single straight-line displacement could the pirate have taken to reach the treasure?

3. Emily passes a soccer ball 6.0 m directly across the field to Kara. Kara then kicks the ball 14.5 m directly down the field to Luisa. What is the total displacement of the ball as it travels between Emily and Luisa?

4. A hummingbird, 3.4 m above the ground, flies 1.2 m along a straight path. Upon spotting a flower below, the hummingbird drops directly downward 1.4 m to hover in front of the flower. What is the hummingbird’s total displacement?

### Alternative Approaches

To find an angle in a right triangle, you can apply different trigonometric functions, such as the following.

\[ \theta = \sin^{-1}\left(\frac{\Delta y}{d}\right) = \sin^{-1}\left(\frac{136}{178}\right) = 49.8^\circ \]
Resolving Vectors into Components

In the pyramid example, the horizontal and vertical parts that add up to give the tourist’s actual displacement are called components. The x component is parallel to the x-axis. The y component is parallel to the y-axis. Any vector can be completely described by a set of perpendicular components.

In this textbook, components of vectors are shown as outlined, open arrows. Components have arrowheads to indicate their direction. Components are scalars (numbers), but they are signed numbers. The direction is important to determine their sign in a coordinate system.

You can often describe an object’s motion more conveniently by breaking a single vector into two components, or resolving the vector. Resolving a vector allows you to analyze the motion in each direction.

This point is illustrated by examining a scene on the set of an action movie. For this scene, a plane travels at 95 km/h at an angle of 20° relative to the ground. Filming the plane from below, a camera team travels in a truck directly beneath the plane, as shown in Figure 2.6.

To find the velocity that the truck must maintain to stay beneath the plane, we must know the horizontal component of the plane’s velocity. Once more, the key to solving the problem is to recognize that a right triangle can be drawn using the plane’s velocity and its x and y components. The situation can then be analyzed using trigonometry.

The sine and cosine functions are defined in terms of the lengths of the sides of such right triangles. The sine of an angle is the ratio of the leg opposite that angle to the hypotenuse.

**Definition of the Sine Function for Right Triangles**

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\text{opposite leg}}{\text{hypotenuse}}
\]

In Figure 2.7, the leg opposite the 20° angle represents the y component, \(v_y\), which describes the vertical speed of the airplane. The hypotenuse, \(v_{plane}\), is the resultant vector that describes the airplane’s total velocity.

The cosine of an angle is the ratio between the leg adjacent to that angle and the hypotenuse.

**Definition of the Cosine Function for Right Triangles**

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{\text{adjacent leg}}{\text{hypotenuse}}
\]

In Figure 2.7, the adjacent leg represents the x component, \(v_x\), which describes the airplane’s horizontal speed. This x component equals the speed required of the truck to remain beneath the plane. Thus, the truck must maintain a speed of \(v_x = (\cos 20°)(95 \text{ km/h}) = 90 \text{ km/h}\).
Resolving Vectors

Sample Problem B  Find the components of the velocity of a helicopter traveling 95 km/h at an angle of 35° to the ground.

1 ANALYZE

Given:  \( v = 95 \text{ km/h} \)  \( \theta = 35^\circ \)
Unknown:  \( v_x = ? \)  \( v_y = ? \)
Diagram:  The most convenient coordinate system is one with the \( x \)-axis directed along the ground and the \( y \)-axis directed vertically.

2 PLAN

Choose an equation or situation:
Because the axes are perpendicular, the sine and cosine functions can be used to find the components.
\[
\sin \theta = \frac{v_y}{v} \\
\cos \theta = \frac{v_x}{v}
\]
Rearrange the equations to isolate the unknowns:
\[
v_y = v \sin \theta \\
v_x = v \cos \theta
\]

3 SOLVE

Substitute the values into the equations and solve:
\[
v_y = (95 \text{ km/h}) (\sin 35^\circ) \\
v_y = 54 \text{ km/h}
\]
\[
v_x = (95 \text{ km/h}) (\cos 35^\circ) \\
v_x = 78 \text{ km/h}
\]

4 CHECK YOUR ANSWER

Because the components of the velocity form a right triangle with the helicopter’s actual velocity, the components must satisfy the Pythagorean theorem.
\[
v^2 = v_x^2 + v_y^2 \\
(95)^2 = (78)^2 + (54)^2 \\
9025 \approx 9000
\]
The slight difference is due to rounding.

ALTERNATIVE APPROACH

For Step 4 (Check Your Answer) students can apply the sine and cosine ratios as shown:
\[
\sin \theta = \frac{v_y}{v} \Rightarrow \sin 35^\circ = \frac{54}{95} \approx 0.573 \approx 0.568
\]
\[
\cos \theta = \frac{v_x}{v} \Rightarrow \cos 35^\circ = \frac{78}{95} \approx 0.819 \approx 0.821
\]
The slight difference is due to rounding.

Classroom Practice

Resolving Vectors  An arrow is shot from a bow at an angle of 25° above the horizontal with an initial speed of 45 m/s. Find the horizontal and vertical components of the arrow’s initial velocity.

Answer: 41 m/s, 19 m/s

The arrow strikes the target with a speed of 45 m/s at an angle of \(-25^\circ\) with respect to the horizontal. Calculate the horizontal and vertical components of the arrow’s final velocity.

Answer: 41 m/s, \(-19\) m/s

PROBLEM GUIDE B

Use this guide to assign problems.
SE = Student Edition Textbook
PW = Sample Problem Set I (online)
PB = Sample Problem Set II (online)

Solving for

One component

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Both components

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*Challenging Problem
Adding Vectors That Are Not Perpendicular

Until this point, the vector-addition problems concerned vectors that are perpendicular to one another. However, many objects move in one direction and then turn at an angle before continuing their motion. Suppose that a plane initially travels 5 km at an angle of 35° to the ground, then climbs at only 10° relative to the ground for 22 km. How can you determine the magnitude and direction for the vector denoting the total displacement of the plane?

Because the original displacement vectors do not form a right triangle, you cannot apply the tangent function or the Pythagorean theorem when adding the original two vectors.

Determining the magnitude and the direction of the resultant can be achieved by resolving each of the plane’s displacement vectors into its x and y components. Then the components along each axis can be added together. As shown in Figure 2.8, these sums will be the two perpendicular components of the resultant, \( d \). The resultant’s magnitude can then be found by using the Pythagorean theorem, and its direction can be found by using the inverse tangent function.

Adding Vectors That Are Not Perpendicular

1. How fast must a truck travel to stay beneath an airplane that is moving 105 km/h at an angle of 25° to the ground?
2. What is the magnitude of the vertical component of the velocity of the plane in item 1?
3. A truck drives up a hill with a 15° incline. If the truck has a constant speed of 22 m/s, what are the horizontal and vertical components of the truck’s velocity?
4. What are the horizontal and vertical components of a cat’s displacement when the cat has climbed 5 m directly up a tree?
Adding Vectors Algebraically

Sample Problem C
A hiker walks 27.0 km from her base camp at 35° south of east. The next day, she walks 41.0 km in a direction 65° north of east and discovers a forest ranger’s tower. Find the magnitude and direction of her resultant displacement between the base camp and the tower.

ANALYZE
Select a coordinate system. Then sketch and label each vector.

Given:
- \( d_1 = 27.0 \text{ km} \)  
- \( \theta_1 = -35° \)  
- \( d_2 = 41.0 \text{ km} \)  
- \( \theta_2 = 65° \)

Unknown:
- \( d = ? \)  
- \( \theta = ? \)

PLAN
Find the \( x \) and \( y \) components of all vectors.

Make a separate sketch of the displacements for each day. Use the cosine and sine functions to find the displacement components.

\[
\begin{align*}
\cos \theta & = \frac{\Delta x}{d} \quad \sin \theta = \frac{\Delta y}{d} \\
\end{align*}
\]

(a) For day 1:
\[\Delta x_1 = d_1 \cos \theta_1 = (27.0 \text{ km}) \cos (-35°) = 22 \text{ km} \]
\[\Delta y_1 = d_1 \sin \theta_1 = (27.0 \text{ km}) \sin (-35°) = -15 \text{ km} \]

(b) For day 2:
\[\Delta x_2 = d_2 \cos \theta_2 = (41.0 \text{ km}) \cos 65° = 17 \text{ km} \]
\[\Delta y_2 = d_2 \sin \theta_2 = (41.0 \text{ km}) \sin 65° = 37 \text{ km} \]

Find the \( x \) and \( y \) components of the total displacement.
\[\Delta x_{tot} = \Delta x_1 + \Delta x_2 = 22 \text{ km} + 17 \text{ km} = 39 \text{ km} \]
\[\Delta y_{tot} = \Delta y_1 + \Delta y_2 = -15 \text{ km} + 37 \text{ km} = 22 \text{ km} \]

SOLVE
Use the Pythagorean theorem to find the magnitude of the resultant vector.
\[
d^2 = (\Delta x_{tot})^2 + (\Delta y_{tot})^2 \]
\[d = \sqrt{(39 \text{ km})^2 + (22 \text{ km})^2} = 45 \text{ km} \]

Use a suitable trigonometric function to find the angle.
\[\theta = \tan^{-1} \left( \frac{\Delta y_{tot}}{\Delta x_{tot}} \right) = \tan^{-1} \left( \frac{22 \text{ km}}{39 \text{ km}} \right) = 29° \text{ north of east} \]
Adding Vectors Algebraically (continued)

Practice

1. A football player runs directly down the field for 35 m before turning to the right at an angle of 25° from his original direction and running an additional 15 m before getting tackled. What is the magnitude and direction of the runner’s total displacement?

2. A plane travels 2.5 km at an angle of 35° to the ground and then changes direction and travels 5.2 km at an angle of 22° to the ground. What is the magnitude and direction of the plane’s total displacement?

3. During a rodeo, a clown runs 8.0 m north, turns 55° north of east, and runs 3.5 m. Then, after waiting for the bull to come near, the clown turns due east and runs 5.0 m to exit the arena. What is the clown’s total displacement?

4. An airplane flying parallel to the ground undergoes two consecutive displacements. The first is 75 km 30.0° west of north, and the second is 155 km 60.0° east of north. What is the total displacement of the airplane?

Reviewing Main Ideas

1. Identify a convenient coordinate system for analyzing each of the following situations:
   a. a dog walking along a sidewalk
   b. an acrobat walking along a high wire
   c. a submarine submerging at an angle of 30° to the horizontal

2. Find the magnitude and direction of the resultant velocity vector for the following perpendicular velocities:
   a. a fish swimming at 3.0 m/s relative to the water across a river that moves at 5.0 m/s
   b. a surfer traveling at 1.0 m/s relative to the water across a wave that is traveling at 6.0 m/s

3. Find the vector components along the directions noted in parentheses.
   a. a car displaced 45° north of east by 10.0 km (north and east)
   b. a duck accelerating away from a hunter at 2.0 m/s² at an angle of 35° to the ground (horizontal and vertical)

Critical Thinking

4. Why do nonperpendicular vectors need to be resolved into components before you can add the vectors together?

Answers to Section Assessment

1. a. x-axis: forward and backward on sidewalk
   y-axis: left and right on sidewalk
   b. x-axis: forward and backward on rope
   y-axis: up and down
   c. x-axis: horizontal at water level
   y-axis: up and down

2. a. 5.8 m/s at 59° downriver from its intended path
   b. 6.1 m/s at 95° from the direction the wave is traveling

3. a. 7.07 km north, 7.07 km east
   b. 1.6 m/s² horizontal, 1.1 m/s² vertical

4. because the Pythagorean theorem and the tangent function can be applied only to right triangles